

Chapman-Enskog method and synchronization of globally coupled oscillators

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The Chapman-Enskog method of kinetic theory is applied to two problems of synchronization of globally coupled phase oscillators. First, a modified Kuramoto model is obtained in the limit of small inertia from a more general model that includes “inertial” effects. Second, a modified Chapman-Enskog method is used to derive the amplitude equation for an $O(2)$ Takens-Bogdanov bifurcation corresponding to the tricritical point of the Kuramoto model with a bimodal distribution of oscillator natural frequencies. This latter calculation shows that the Chapman-Enskog method is a convenient alternative to normal form calculations.

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I. INTRODUCTION

The Chapman-Enskog method (CEM) has long been used to derive hydrodynamic equations of parabolic type from kinetic equations of the Boltzmann type [1,2]. The complexity of these equations has perhaps hindered the realization that the CEM is a powerful singular perturbation method that can be used advantageously as a viable alternative to multiple scales or normal form calculations in different contexts. Motivated by recent work on the subject of models of synchronization of phase oscillators [3–5], I shall derive here reduced equations for two different Kuramoto models by means of the CEM.

The first example corresponds to the limit of small inertia in a generalized Kuramoto model of globally coupled phase oscillators [6]:

$$\begin{aligned}\dot{\theta}_j &= \omega_j, \\ m\dot{\omega}_j &= -\omega_j + \Omega_j + Kr_N \sin(\psi_N - \theta_j) + \xi_j(t), \\ j &= 1, \dots, N,\end{aligned}\quad (1)$$

where θ_j , ω_j , and Ω_j denote the phase, frequency, and natural frequency of the j th oscillator, respectively. The natural frequencies are distributed with probability density $g(\Omega)$, which may have a single maximum (*unimodal* distribution) or several peaks (*multimodal* distribution). The positive parameters m and K are the “inertia” and the coupling strength, respectively. The complex order parameter defined by

$$r_N e^{i\psi_N} = \frac{1}{N} \sum_{j=1}^N e^{i\theta_j} \quad (2)$$

measures phase synchronization: in the limit as $N \rightarrow \infty$, $r_N > 0$ if the oscillators are synchronized and $r_N = 0$ if not. Finally, the ξ_j 's are independent identically distributed Gaussian white noises, with $\langle \xi_j \rangle = 0$, $\langle \xi_i(t) \xi_j(s) \rangle = 2D \delta_{ij} \delta(t-s)$. Without white noise terms, these equations were proposed to account for synchronization of biological systems [7,8]. When the inertial terms vanish, $m=0$, Eqs. (1) and (2) are

exactly the usual Kuramoto model [9,10]. An important technologically relevant application of this model is the study of superconducting Josephson junction arrays [11,12]. It has been shown that the usual Kuramoto model (with $m=0$) describes series arrays in the limiting case of weak coupling and disorder of arrays with zero junction capacitance [11]. A currently open problem is to derive a reduced model for the averaged system in the limiting case of weak coupling, weak disorder, and nonzero junction capacitance. In so far as capacitance plays the same role as inertia in simple electrical networks, the Kuramoto model with inertia might be useful to understand the effect of nonzero electrical capacitance in Josephson junction arrays.

In the limiting case of infinitely many oscillators, models with mean-field coupling are described by an evolution equation for the one-oscillator probability density $\rho(\theta, \omega, \Omega, t)$ [13]. For the present model this equation is [6]

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= \frac{D}{m^2} \frac{\partial^2 \rho}{\partial \omega^2} - \frac{1}{m} \frac{\partial}{\partial \omega} \{ [-\omega + \Omega + Kr \sin(\psi - \theta)] \rho \} \\ &\quad - \omega \frac{\partial \rho}{\partial \theta},\end{aligned}\quad (3)$$

where the order parameter is now given by

$$r e^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\theta} \rho(\theta, \omega, \Omega, t) g(\Omega) d\Omega d\omega d\theta. \quad (4)$$

Equations (3) and (4) should be supplemented with appropriate initial and boundary data (ρ is 2π periodic in θ and has suitable decay behavior as $\omega \rightarrow \pm\infty$) plus the normalization condition

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} \rho(\theta, \omega, \Omega, t) d\omega d\theta = 1. \quad (5)$$

An extensive study of oscillator synchronization in the model (3)–(5) was carried out in a previous paper [5] by using different analytical and numerical techniques. The limiting case of small inertia, $m \rightarrow 0$, was analyzed by an arbitrary three-mode truncation of a mode-coupling expansion of ρ [5]. It is rather intriguing that exactly the same small inertia results had been obtained earlier by Hong *et al.* [4] using completely different methods. Hong *et al.* obtained first a reduced Smoluchowski type equation, and then analyzed this reduced equation. The derivation of the reduced equation

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consisted of truncating a moment hierarchy for the Fokker-Planck equation, as explained by Schneider *et al.* [14]. Both procedures yielded correctly the linear term of the amplitude equation describing the synchronization transition, but not the cubic term in this equation. In this paper, we shall obtain a consistent reduced Smoluchowski type equation by using the CEM.

Our second example corresponds to the particular case of setting $m=0$ and $g(\Omega)=1/2[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$ in Eq. (1). Then the phase diagram of the incoherent probability density $\rho=1/(2\pi)$ contains a tricritical point where a line of Hopf bifurcations intersects tangentially a line of homoclinic orbits at zero frequency. The resulting Takens-Bogdanov bifurcation with $O(2)$ symmetry was analyzed in [3] by multiple scale techniques [15]. A shortcoming of these techniques is that all terms in the corresponding amplitude equation are necessarily of the same order, while terms of different order are needed to describe the Takens-Bogdanov point. We shall show here how to adapt multiple scale ideas using the CEM to overcome this difficulty. Although the resulting amplitude equation is known, the calculation presented here is of independent interest to illustrate the possible applications of our presentation of the CEM to deriving amplitude and bifurcation equations containing different asymptotic orders.

The rest of the paper is as follows. The Smoluchowski equation for the small inertia limit of the Kuramoto model with inertia is derived in Sec. II. Section III contains the application of the CEM to finding bifurcation equations. After explaining how the procedure works for the well-known Hopf bifurcation, we describe how to obtain the normal form corresponding to an $O(2)$ -symmetric Takens-Bogdanov bifurcation for the bimodal Kuramoto model (without inertia). The last section contains our conclusions.

II. CHAPMAN-ENSKOG METHOD FOR THE GENERALIZED KURAMOTO MODEL IN THE LIMITING CASE OF SMALL INERTIA

To find a reduced equation, we should first nondimensionalize Eqs. (3) and (4) in an appropriate way. The main idea is that the force terms and diffusion in velocity space should be dominant. Then the probability density ρ rapidly reaches local equilibrium in velocity and its slowly varying amplitude obeys a Smoluchowski equation. This means that Ω and K should have the same order as ω , and that the terms $m^{-1}(\omega\rho)_\omega$ and $(D/m^2)\rho_{\omega\omega}$ should be of the same order (subscripts mean the partial derivative with respect to the corresponding variable). If we call ω_0 a typical unit of velocity, the latter balance yields the following velocity scale:

$$\omega_0 = \sqrt{\frac{D}{m}}, \quad (6)$$

which is just the thermal velocity. The ratio of $\omega\rho_\theta$ to $m^{-1}(\omega\rho)_\omega$ is of the order

$$\epsilon = \sqrt{mD}, \quad (7)$$

which will be our small dimensionless parameter. Lastly, we shall choose the time unit so that ρ_t and $\omega\rho_\theta$ are of the same order. This choice is dictated by the mechanics of the CEM (see below) and yields a time unit $t_0=1/\omega_0$. The normaliza-

tion condition and the definition of the order parameter dictate that $g(\Omega)$ and ρ are to be measured in units of $1/\omega_0$ too. The angle θ is already dimensionless.

Recapitulating, ω , Ω , and K are measured in units of the thermal velocity ω_0 , while ρ , $g(\Omega)$, and time are measured in units of the reciprocal velocity $1/\omega_0$. Then the dimensionless Fokker-Planck equation is

$$\frac{\partial}{\partial\omega} \left\{ \frac{\partial\rho}{\partial\omega} + [\omega + Kr \sin(\theta - \psi) - \Omega] \rho \right\} = \epsilon \left(\frac{\partial\rho}{\partial t} + \omega \frac{\partial\rho}{\partial\theta} \right), \quad (8)$$

to be solved together with Eqs. (4) and (5) and appropriate periodicity, initial, and decay conditions as $\omega \rightarrow \pm\infty$.

A different ‘‘parabolic’’ scaling is usual in kinetic theory: in it the term $[Kr \sin(\theta - \psi) - \Omega]\rho_\omega$ is $O(\epsilon)$ and the time derivative is $O(\epsilon^2)$. Using the CEM with this scaling yields the standard Kuramoto model with $m=0$ as a leading order approximation, while higher order modifications contain derivatives of order 4 in θ and higher. We think the ‘‘hyperbolic scaling’’ (8) will yield results that are valid over a much larger range of independent variables and of parameters.

Chapman-Enskog method

Setting $\epsilon=0$ in Eq. (8), we find a simple equation to be solved together with Eqs. (4) and (5). Its solution is a displaced Maxwellian:

$$\rho = \frac{e^{-v^2/2}}{\sqrt{2\pi}} P(\theta, \Omega, t), \quad (9)$$

$$V = \omega + Kr \sin(\theta - \psi) - \Omega, \quad (10)$$

$$r = \int_0^{2\pi} \int_{-\infty}^{\infty} e^{i(\theta - \psi)} P(\theta, \Omega, t) d\theta g(\Omega) d\Omega, \quad (11)$$

$$\int_0^{2\pi} P(\theta, \Omega, t) d\theta = 1. \quad (12)$$

Notice that $P(\theta, \Omega, t)$ is an arbitrary function of θ and t except in Eq. (12). Furthermore, Eq. (9) correspond to a particular form of the initial conditions. The Chapman-Enskog ansatz consists of assuming that ρ has the following asymptotic expansion:

$$\rho = \frac{e^{-v^2/2}}{\sqrt{2\pi}} P(\theta, \Omega, t; \epsilon) + \sum_{n=1}^{\infty} \epsilon^n \rho^{(n)}(\theta, \omega, \Omega; P). \quad (13)$$

Furthermore, we impose the condition that the amplitude P obeys the equation

$$\frac{\partial P}{\partial t} = \sum_{n=0}^{\infty} \epsilon^n F^{(n)}(P), \quad (14)$$

where $F^{(n)}$ are functionals of P to be determined as the procedure goes on. This equation for P is not explicitly written in the usual presentations of the CEM [1,2]. Instead, the form of this equation is guessed by writing equations for the moments of ρ and using gradient expansions. We find this latter procedure more confusing.

Insertion of Eqs. (13) and (14) into the equations and auxiliary conditions yields a hierarchy of linear equations for the $\rho^{(n)}$. Notice that the latter depend on time only through their dependence on P . The functionals $F^{(n)}(P)$ are determined so that each equation (and set of auxiliary conditions) for $\rho^{(n)}$ has a solution that is bounded for all values of ω , even as $\omega \rightarrow \pm\infty$. Once a sufficient number of $F^{(n)}$ is determined, Eq. (14) is the sought amplitude equation. Please notice that, unlike results from the method of multiple scales, terms in Eq. (14) may be of different order.

Let us illustrate how the procedure works by finding $F^{(0)}$ and $F^{(1)}$. Insertion of Eqs. (13) and (14) in Eqs. (8), (4), and (5) yields the following hierarchy of linear equations:

$$\begin{aligned} \mathcal{L}\rho^{(1)} = & \frac{\omega e^{-V^2/2}}{\sqrt{2\pi}} [P_\theta - VPKr^{(0)} \cos(\theta - \psi)] \\ & + \frac{e^{-V^2/2}}{\sqrt{2\pi}} \{F^{(0)} - VPK[r^{(0)} \sin(\theta - \psi) \\ & - r^{(0)} \dot{\psi} \cos(\theta - \psi)]\}, \end{aligned} \quad (15)$$

$$\begin{aligned} \mathcal{L}\rho^{(2)} = & \frac{e^{-V^2/2}}{\sqrt{2\pi}} F^{(1)} + Kr^{(1)} \sin(\theta - \psi) \rho_\omega^{(1)} \\ & + \omega \rho_\theta^{(1)} + \rho_t^{(1)}|_{P_t=F^{(0)}}, \end{aligned} \quad (16)$$

and so on. These equations are to be supplemented by the normalization conditions (12) and

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \rho^{(n)} d\theta d\omega = 0, \quad (17)$$

and the definitions

$$r^{(n)} = \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\theta - \psi)} \rho^{(n)} d\theta d\omega g(\Omega) d\Omega, \quad (18)$$

$$\mathcal{L}\rho^{(n)} = [V\rho^{(n)} + \rho_\omega^{(n)}]_\omega - \frac{Ve^{-V^2/2}}{\sqrt{2\pi}} Kr^{(n)} \sin(\theta - \psi), \quad (19)$$

for $n=1, 2, \dots, V$ is now given by Eq. (10) with $r=r^{(0)}$ given by Eq. (11). Lastly, $\dot{r}^{(0)}$ and $\dot{\psi}$ are calculated by taking the time derivative of Eq. (11) and using the first term in Eq. (14) to replace P_t by $F^{(0)}$.

Let us now consider Eq. (15). Since the ω integral of its left hand side is zero, this equation has a solution only if the ω integral of its right hand side is zero. The corresponding integrals are simplified by using the symmetry of the Maxwellian and shifting integration variables from ω to V . The vanishing of the integral of the right side yields

$$F^{(0)} = \{[Kr^{(0)} \sin(\theta - \psi) - \Omega]P\}_\theta. \quad (20)$$

Notice that we needed $F^{(0)}$ in the right side of Eq. (15) for this equation to have an appropriate solution. In turn, $F^{(0)}$ appeared in this equation due to our choice of the time unit. Thus the time unit is dictated by the solvability conditions of the hierarchy of equations generated by the CEM.

Inserting now Eq. (20) in Eq. (15), we realize that the right hand side thereof is the partial derivative of some expression with respect to ω . This greatly helps in finding the solution

$$\begin{aligned} \rho^{(1)} = & \frac{e^{-V^2/2}}{\sqrt{2\pi}} \left(\frac{V^2 - 1}{2} PKr^{(0)} \cos(\theta - \psi) \right. \\ & + V(KP\dot{r}^{(0)} \sin(\theta - \psi) - r^{(0)} \dot{\psi} \cos(\theta - \psi) \\ & \left. - [Kr^{(0)} \sin(\theta - \psi) - \Omega]r^{(0)} \cos(\theta - \psi) - P_\theta \right), \end{aligned} \quad (21)$$

which satisfies Eq. (17) and yields $r^{(1)}=0$. Notice that a term of the form (9) (satisfying $\int_0^{2\pi} P d\theta=0$) could have been added to the solution (21). However, all such terms are already contained in the ansatz (13) with $P=P(\theta, \Omega, t; \epsilon)$, and we shall therefore omit them.

To find $F^{(1)}$, we insert Eq. (21) in Eq. (16) and use the solvability condition for this equation. Simplifications arise from the identities

$$\int_{-\infty}^{\infty} \rho_t^{(1)} d\omega = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho^{(1)} d\omega = 0$$

and

$$\int_{-\infty}^{\infty} \omega \rho_\theta^{(1)} d\omega = \frac{\partial}{\partial \theta} \int_{-\infty}^{\infty} \omega \rho^{(1)} d\omega.$$

The result is

$$\begin{aligned} F^{(1)} = & \frac{\partial}{\partial \theta} (P_\theta + \{[Kr^{(0)} \sin(\theta - \psi) - \Omega + \dot{\psi}]Kr^{(0)} \cos(\theta - \psi) \\ & - Kr^{(0)} \sin(\theta - \psi)\}P). \end{aligned} \quad (22)$$

In this expression, $\dot{r}^{(0)}$ and $\dot{\psi}$ can be found from their definitions and Eq. (20). Integration by parts yields

$$\dot{r}^{(0)} + ir^{(0)} \dot{\psi} = \frac{Kr^{(0)}}{2} (1 - \langle e^{2i(\theta - \psi)} \rangle) + i \langle \Omega e^{i(\theta - \psi)} \rangle \Rightarrow$$

$$\dot{r}^{(0)} = Kr^{(0)} \langle \sin^2(\theta - \psi) \rangle - \langle \Omega \sin(\theta - \psi) \rangle,$$

$$r^{(0)} \dot{\psi} = \langle \Omega \cos(\theta - \psi) \rangle - \frac{Kr^{(0)}}{2} \langle \sin[2(\theta - \psi)] \rangle, \quad (23)$$

where

$$\langle f(\theta, \Omega) \rangle \equiv \int_0^{2\pi} \int_{-\infty}^{\infty} f(\theta, \Omega) P(\theta, \Omega, t; \epsilon) d\theta g(\Omega) d\Omega. \quad (24)$$

We can now insert Eqs. (20), (22), and (23) into Eq. (14) to obtain the sought Smoluchowski equation for P :

$$\begin{aligned}
P_t - \frac{\partial}{\partial \theta} \left\{ \left[[Kr^{(0)} \sin(\theta - \psi) - \Omega][1 + \epsilon Kr^{(0)} \cos(\theta - \psi)] \right. \right. \\
- \epsilon K \cos(\theta - \psi) \left(\frac{Kr^{(0)}}{2} \langle \sin[2(\theta - \psi)] \rangle \right. \\
- \langle \Omega \cos(\theta - \psi) \rangle \left. \right) - \epsilon K \sin(\theta - \psi) [Kr^{(0)} \langle \sin^2(\theta - \psi) \rangle \\
\left. \left. - \langle \Omega \sin(\theta - \psi) \rangle \right] P + \epsilon P_\theta \right\} = 0. \quad (25)
\end{aligned}$$

Restoring the original dimensional units, this equation becomes

$$\begin{aligned}
P_t - \frac{\partial}{\partial \theta} \left\{ \left[[Kr \sin(\theta - \psi) - \Omega][1 + mKr \cos(\theta - \psi)] \right. \right. \\
- mK \cos(\theta - \psi) \left(\frac{Kr}{2} \langle \sin[2(\theta - \psi)] \rangle \right. \\
- \langle \Omega \cos(\theta - \psi) \rangle \left. \right) - mK \sin(\theta - \psi) [Kr \langle \sin^2(\theta - \psi) \rangle \\
\left. \left. - \langle \Omega \sin(\theta - \psi) \rangle \right] P + DP_\theta \right\} = 0. \quad (26)
\end{aligned}$$

to be solved together with Eq. (11), definition (24), normalization condition (12), 2π -periodicity in θ , and appropriate initial conditions.

Equation (26) is the main result of this section. If $m=0$, the usual Kuramoto model for phase oscillators is recovered. Comparing this result with Eq. (5) of Hong *et al.*, we see that the second and third terms proportional to m in the drift current were missing and that their equation contained an additional diffusive term. The latter would be order ϵ^2 in our scheme. These differences can be tracked to the fact that Hong *et al.* followed a previous paper [14], whose authors used a moment method with arbitrary closure assumptions to derive the Smoluchowski equation.

III. DERIVATION OF BIFURCATION EQUATIONS BY THE CHAPMAN-ENSKOG METHOD

First of all, we shall illustrate the application of the CEM to a well-known case: deriving the amplitude equation for the usual Hopf bifurcation. Next the much more complicated calculation of the amplitude equation for the tricritical point of the bimodal Kuramoto model will be tackled.

A. The usual Hopf bifurcation

Let us consider a system of n ordinary differential equations for an unknown n -component vector $u(t; \alpha)$:

$$\frac{du}{dt} = f(u; \alpha); \quad (27)$$

α is the bifurcation parameter. We shall assume that $u=0$ is a linearly stable solution if $\alpha < 0$ and that it is unstable if $\alpha > 0$. All eigenvalues of $L(\alpha) \equiv \partial f(0, \alpha) / \partial u$ except $\lambda(\alpha)$ and its complex conjugate have negative real parts in the neighborhood of $\alpha=0$. We have $\lambda(\alpha) \sim i\omega_0 + \alpha\lambda'(0)$ as $\alpha \rightarrow 0$, with $\text{Re } \lambda'(0) > 0$. The linearized equation of (27) about $u=0$, $\alpha=0$ has the solution $u = A\phi_0 e^{i\omega_0 t} + \text{c.c.}$, plus

terms that decay exponentially fast to zero as time increases. ϕ_0 is the eigenvector corresponding to $i\omega_0$ and A is a complex constant.

To apply the CEM to this problem, we shall assume

$$\alpha = \alpha_1 \epsilon + \alpha_2 \epsilon^2 + O(\epsilon^3),$$

$$u = \epsilon A(t; \epsilon) \phi_0 e^{i\omega_0 t} + \text{c.c.} + \sum_{n=2}^{\infty} \epsilon^n u^{(n)}(t; A, \bar{A}), \quad (28)$$

where the amplitude A obeys the ansatz

$$\frac{dA}{dt} = \sum_{n=1}^{\infty} \epsilon^n F^{(n)}(A, \bar{A}). \quad (29)$$

Here \bar{A} is the complex conjugate of A and c.c. means the complex conjugate of the preceding term. The parameter ϵ measures the size of the bifurcation amplitude and its relation to the bifurcation parameter α is determined by finding the coefficients α_j . Notice that A varies slowly with time (on a scale $\epsilon^2 t$, as we will see). The functions $u^{(n)}$ depend on a fast scale t corresponding to stable exponentially decaying modes, and on a slow time scale through their dependence on A . All terms in Eq. (28) that decrease exponentially in time will be omitted. α_n and $F^{(n)}$ are determined so that the solutions $u^{(n)}$ are bounded as $t \rightarrow \infty$ (on the fast time scale), for fixed A . Substitution of Eq. (28) and (29) in Eq. (27) yields the following hierarchy of linear equations:

$$[L(0) - i\omega_0] \phi_0 = 0,$$

$$\begin{aligned}
\frac{du^{(2)}}{dt} - L(0)u^{(2)} = -F^{(1)}\phi_0 e^{i\omega_0 t} + \text{c.c.} + \alpha_1 L'(0)\phi_0 A e^{i\omega_0 t} \\
+ \text{c.c.} + \frac{1}{2} f_{uu}(0; 0) : (A\phi_0 e^{i\omega_0 t} + \text{c.c.})^2, \quad (30)
\end{aligned}$$

$$\begin{aligned}
\frac{du^{(3)}}{dt} - L(0)u^{(3)} = \alpha_1(\dots) - F^{(2)}\phi_0 e^{i\omega_0 t} + \text{c.c.} \\
+ \alpha_2 L'(0)\phi_0 A e^{i\omega_0 t} + \text{c.c.} \\
+ \frac{1}{2} f_{uu}(0; 0) : u^{(2)}(A\phi_0 e^{i\omega_0 t} + \text{c.c.}) \\
+ \frac{1}{6} f_{uuu}(0; 0) : (A\phi_0 e^{i\omega_0 t} + \text{c.c.})^3, \quad (31)
\end{aligned}$$

and so on. Here $f_{uu}(0; 0)$ is an $n \times n$ matrix and the colon means tensor contraction, etc. The first equation holds identically due to the definitions of ϕ_0 and $i\omega_0$. The other equations should be solved for bounded $u^{(n)}$ as $t \rightarrow \infty$. Their solutions should not contain terms of the form $B\phi_0 e^{i\omega_0 t} + \text{c.c.}$, solving the corresponding linear homogeneous problem. The reason for this latter requirement is that all such terms are already contained in $A(t; \epsilon)$.

Equation (30) yields

$$\alpha_1 = 0, \quad F^{(1)} = 0,$$

$$\begin{aligned}
u^{(2)} = \frac{A^2}{2} e^{i2\omega_0 t} [2i\omega_0 I - L(0)]^{-1} f_{uu}(0; 0) : \phi_0^2 \\
+ \text{c.c.} - |A|^2 L(0)^{-1} f_{uu}(0; 0) : \phi_0 \bar{\phi}_0. \quad (32)
\end{aligned}$$

(I is the identity matrix of order n .) Solvability of Eq. (31) yields

$$F^{(2)} = \alpha_2 \lambda'(0)A - \mu A |A|^2, \quad (33)$$

where

$$\lambda'(0) = \frac{\langle \phi_0^\dagger, L'(0)\phi_0 \rangle}{\langle \phi_0^\dagger, \phi_0 \rangle}, \quad (34)$$

$$\begin{aligned} \langle \phi_0^\dagger, \phi_0 \rangle \mu &= \frac{1}{2} \langle \phi_0^\dagger, f_{uu}(0;0) : \overline{\phi_0} \\ &\times [L(0) - i2\omega_0 I]^{-1} f_{uu}(0;0) : \phi_0^2 \rangle \\ &+ \langle \phi_0^\dagger, f_{uu}(0;0) : \phi_0 L(0)^{-1} f_{uu}(0;0) : \phi_0 \overline{\phi_0} \rangle \\ &- \frac{1}{2} \langle \phi_0^\dagger, f_{uuu}(0;0) : \phi_0^2 \overline{\phi_0} \rangle. \end{aligned} \quad (35)$$

Here ϕ_0^\dagger is the eigenvector corresponding to the adjoint problem

$$L^\dagger(0)\phi_0^\dagger = i\omega_0\phi_0^\dagger.$$

Substituting these results in Eq. (29), the following amplitude equation is found:

$$\frac{dA}{dt} = \varepsilon^2 \alpha_2 \lambda'(0)A - \varepsilon^2 \mu A |A|^2 + O(\varepsilon^3). \quad (36)$$

These formulas for the bifurcation equation and its coefficients have been obtained many times before; see [16] for an equivalent explicit determination of μ , whose real part decides whether the bifurcation is sub- or supercritical. If $\text{Re}\mu = 0$, we should calculate higher order terms in the Chapman-Enskog expansion (29). The systematic way in which the CEM yields such terms is a great advantage with respect to other methods such as multiple scales [15].

Similar ideas can be used to derive amplitude equations for pattern forming systems governed by partial differential equations [17]. In such cases, we have to rescale space variables appropriately and assume that the $F^{(n)}$ also depend on spatial derivatives of A .

B. Symmetric Takens-Bogdanov bifurcation

Our starting point in this subsection is the standard Kuramoto model with bimodal natural frequency distribution, i.e., Eq. (26) with $m=0$ and $g(\Omega) = \frac{1}{2} [\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$. The phase diagram of the incoherent solution $P = 1/(2\pi)$ was depicted in Fig. 1 of [3]. At the tricritical point $P = (K/D=4, \Omega_0/D=1)$ a branch of Hopf bifurcations coalesces with a branch with stationary bifurcations and a branch of homoclinic orbits, in an $O(2)$ -symmetric Takens-Bogdanov bifurcation point. A method of multiple scales was used in [3] to analyze this complicated bifurcation. This method was not completely satisfactory because it led to two amplitude equations whose solutions were later interpreted as the two terms of a normal form expansion [3]. Let us show how a modified CEM leads directly to the amplitude equation. Near the tricritical point, we may define an expansion parameter ε so that

$$K = K_c + K_2 \varepsilon^2 + O(\varepsilon^3), \quad \Omega_0 = \Omega_{0c} + \Omega_2 \varepsilon^2 + O(\varepsilon^3) \\ (K_c = 4D, \Omega_{0c} = D). \quad (37)$$

The basic slow time scale near the tricritical point is $T = \varepsilon t$, and the method of multiple scales shows that resonant terms appear in the equations of order ε^3 and higher [3]. Borrowing from the results of that reference (which includes making an exponential ansatz for P), we shall make the following Chapman-Enskog ansatz:

$$P(\theta, \Omega, t; \varepsilon) = \frac{1}{2\pi} \exp\left(\frac{A(T; \varepsilon)}{D + i\Omega} e^{i\theta} + \text{c.c.} \right. \\ \left. + \sum_{j=2}^4 \varepsilon^j \sigma_j(\theta, t, T; A) + O(\varepsilon^5) \right), \quad (38)$$

$$A_{TT} = F^{(0)}(A) + \varepsilon F^{(1)}(A) + O(\varepsilon^2). \quad (39)$$

Terms that decay exponentially on the fast time scale t will be systematically omitted. The equation for A is second order [not first order like Eq. (29)] because resonant terms appear at $O(\varepsilon^3)$ for the first time and they are proportional to A_{TT} . Inserting these equations in Eq. (26) (with $m=0$), we obtain the following hierarchy of linear equations:

$$\mathcal{L}\left(\sigma_2 + \frac{\sigma_1^2}{2}\right) = -4D \partial_\theta (\sigma_1 \text{Im} e^{-i\theta} \langle e^{i\theta'}, \sigma_1 \rangle) - \frac{A_T e^{i\theta}}{D + i\Omega} + \text{c.c.}, \quad (40)$$

$$\int_0^{2\pi} \left(\sigma_2 + \frac{\sigma_1^2}{2}\right) d\theta = 0,$$

$$\begin{aligned} \mathcal{L}\left(\sigma_3 + \sigma_1 \sigma_2 + \frac{\sigma_1^3}{6}\right) &= -4D \partial_\theta \left[\sigma_1 \text{Im} e^{-i\theta} \left\langle e^{i\theta'}, \sigma_2 + \frac{\sigma_1^2}{2} \right\rangle + \left(\sigma_2 + \frac{\sigma_1^2}{2}\right) \text{Im} e^{-i\theta} \langle e^{i\theta'}, \sigma_1 \rangle + \Omega_2 \text{Im} e^{-i\theta} \langle e^{i\theta'}, \sigma_1 \rangle' \right] \\ &- K_2 \partial_\theta \text{Im} e^{-i\theta} \langle e^{-i\theta'}, \sigma_1 \rangle - \partial_T \left(\sigma_2 + \frac{\sigma_1^2}{2}\right), \end{aligned} \quad (41)$$

$$\int_0^{2\pi} \left(\sigma_3 + \sigma_1 \sigma_2 + \frac{\sigma_1^3}{6}\right) d\theta = 0,$$

$$\begin{aligned}
\mathcal{L}\left(\sigma_4 + \sigma_1\sigma_3 + \frac{\sigma_2^2}{2} + \frac{\sigma_1^2\sigma_2}{2} + \frac{\sigma_1^4}{4!}\right) &= -4D\partial_\theta\left[\sigma_1 \operatorname{Im} e^{-i\theta}\left\langle e^{-i\theta'}, \sigma_3 + \sigma_1\sigma_2 + \frac{\sigma_1^3}{6}\right\rangle + \left(\sigma_3 + \sigma_1\sigma_2 + \frac{\sigma_1^3}{6}\right) \operatorname{Im} e^{-i\theta}\langle e^{i\theta'}, \sigma_1\rangle\right. \\
&\quad + \Omega_2 \operatorname{Im} e^{-i\theta}\left\langle e^{i\theta'}, \sigma_2 + \frac{\sigma_1^2}{2}\right\rangle' + \Omega_2\sigma_1 \operatorname{Im} e^{i\theta}\langle e^{i\theta'}, \sigma_1\rangle' + \left(\sigma_2 + \frac{\sigma_1^2}{2}\right) \\
&\quad \times \operatorname{Im} e^{-i\theta}\left\langle e^{i\theta'}, \sigma_2 + \frac{\sigma_1^2}{2}\right\rangle\left. - K_2\partial_\theta\left(\operatorname{Im} e^{-i\theta}\left\langle e^{i\theta'}, \sigma_2 + \frac{\sigma_1^2}{2}\right\rangle\right)\right. \\
&\quad \left. + \sigma_1 \operatorname{Im} e^{-i\theta}\langle e^{i\theta'}, \sigma_1\rangle\right) - \partial_T\left(\sigma_3 + \sigma_1\sigma_2 + \frac{\sigma_1^3}{2}\right), \\
\int_0^{2\pi}\left(\sigma_4 + \sigma_1\sigma_3 + \frac{\sigma_2^2}{2} + \frac{\sigma_1^2\sigma_2}{2} + \frac{\sigma_1^4}{4!}\right)d\theta &= 0.
\end{aligned} \tag{42}$$

Here

$$\mathcal{L}\sigma_n = (\partial_t - D\partial_\theta^2 + \Omega\partial_\theta)\sigma_n + 4D\partial_\theta(\operatorname{Im} e^{-i\theta}\langle e^{i\theta'}, \sigma_n\rangle), \tag{43}$$

and we have used the abbreviations $\sigma_1 = A(T; \varepsilon)e^{i\theta}/(D + i\Omega) + \text{c.c.}$ and

$$\begin{aligned}
\langle \alpha(\theta, \Omega), \beta(\theta, \Omega) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{+\infty} \alpha(\theta, \Omega)\beta(\theta, \Omega)g(\Omega)d\theta d\Omega, \\
\langle \alpha(\theta, \Omega), \beta(\theta, \Omega) \rangle' &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{+\infty} \alpha(\theta, \Omega)\beta(\theta, \Omega)g'_{\Omega_0}(\Omega)d\theta d\Omega,
\end{aligned} \tag{44}$$

$$\begin{aligned}
\langle \alpha(\theta, \Omega), \beta(\theta, \Omega) \rangle' &= \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{+\infty} \alpha(\theta, \Omega)\beta(\theta, \Omega)g'_{\Omega_0}(\Omega)d\theta d\Omega, \\
\end{aligned} \tag{45}$$

where

$$g'_{\Omega_0}(\Omega) = \frac{1}{2}[\delta'(\Omega + \Omega_0) - \delta'(\Omega - \Omega_0)](\Omega_0 = \Omega_{0c} = D). \tag{46}$$

The ansatz (39) has not yet been inserted in Eqs. (40) and (42) in order not to complicate these equations further. We shall keep in mind that these equations will have to be modified later when the solution of Eq. (40) is inserted in Eqs. (41) and (42). The linear equations in the preceding hierarchy should have solutions periodic in θ . A solution of the homogeneous equation $\mathcal{L}U=0$ could be added to the solutions of the linear nonhomogeneous equations in the preceding hierarchy. However, all such terms are to be omitted, for the amplitude $A(T; \varepsilon)$ already takes care of them.

The solution of Eq. (40),

$$\mathcal{L}\left(\sigma_2 + \frac{\sigma_1^2}{2}\right) = -\frac{A_T}{D + i\Omega}e^{i\theta} + \text{c.c.} + \frac{2A^2}{D + i\Omega}e^{2i\theta} + \text{c.c.}, \tag{47}$$

is

$$\begin{aligned}
\sigma_2 + \frac{\sigma_1^2}{2} &= -\frac{A_T}{(D + i\Omega)^2}e^{i\theta} + \text{c.c.} + \frac{A^2}{(D + i\Omega)(2D + i\Omega)}e^{2i\theta} \\
&\quad + \text{c.c.}
\end{aligned} \tag{48}$$

Note that one term proportional to $e^{i\theta}/(D + i\Omega)$ and other terms decaying on the fast scale t are solutions of $\mathcal{L}U=0$ and could have been added to Eq. (48). According to what was said above, all such terms are to be omitted.

We now insert Eq. (48) in Eqs. (41) and (42) together with the CE ansatz (39). The solution of Eq. (41) is

$$\begin{aligned}
\sigma_3 + \sigma_1\sigma_2 + \frac{\sigma_1^3}{6} &= \left(\frac{K_2 - 4\Omega_2}{4D(D + i\Omega)}A + \frac{F^{(0)}}{(D + i\Omega)^3}\right. \\
&\quad \left. - \frac{A|A|^2}{(D + i\Omega)^2(2D + i\Omega)}\right)e^{i\theta} + \text{c.c.} \\
&\quad - \frac{AA_T[1/(D + i\Omega) + 1/(2D + i\Omega)]}{(D + i\Omega)(2D + i\Omega)}e^{2i\theta} \\
&\quad + \text{c.c.} + \frac{A^3e^{3i\theta}}{(D + i\Omega)(2D + i\Omega)(3D + i\Omega)} \\
&\quad + \text{c.c.}
\end{aligned} \tag{49}$$

From this we obtain σ_3 , and finally, from Eq. (42), σ_4 . To obtain the leading order approximation, we only need to determine $A(T; \varepsilon)$. Now, Eq. (49) holds provided that the *non-resonance condition* (needed to remove secular terms)

$$\left\langle \frac{1}{D + i\Omega}, P(\Omega, T; A) \right\rangle = 0 \tag{50}$$

holds, where $P(\Omega, T; A)$ denotes the coefficient of $e^{i\theta}$ on the right hand side of Eq. (41) [3]. Equation (50) yields

$$F^{(0)} = \frac{D}{2}(K_2 - 4\Omega_2)A + \frac{2}{5}|A|^2A. \tag{51}$$

The function $F^{(1)}$ is determined from a similar nonresonance condition for Eq. (42): the coefficient $Q(\Omega, T; A)$ of $e^{i\theta}$ on the right hand side of Eq. (42) should also satisfy Eq. (50). The result is

$$F^{(1)} = \frac{K_2}{2} A_T - \frac{(|A|^2 A)_T}{5D} - \frac{23}{25D} |A|^2 A_T. \quad (52)$$

Insertion of Eqs. (51) and (52) into Eq. (39) yields the sought amplitude equation:

$$\begin{aligned} A_{TT} - \frac{D}{2} (K_2 - 4\Omega_2) A - \frac{2}{5} |A|^2 A \\ = \varepsilon \left(\frac{K_2}{2} A_T - \frac{23}{25D} |A|^2 A_T - \frac{1}{5D} (|A|^2 A)_T \right) + O(\varepsilon^2). \end{aligned} \quad (53)$$

This equation is the scaled ‘‘normal form’’ studied by Dangelmayr and Knobloch in [18] [cf. their Eqs. (3.3), p. 2480]. The general analysis developed in that reference for general scaled normal forms was employed in [3] to study Eq. (53) and will not be repeated here.

IV. CONCLUSIONS

We have applied a modified Chapman-Enskog method to two problems related to Kuramoto models of synchronization of globally coupled phase oscillators. First of all, we found a consistent two-term Smoluchowski approximate equation for a model of oscillators with inertia in the limit of small inertia (as $mD \rightarrow 0+$). Second, we modified the

Chapman-Enskog method to find directly the scaled normal form corresponding to the O(2)-symmetric Takens-Bogdanov bifurcation at the tricritical point of a standard Kuramoto model with bimodal distribution of oscillator natural frequencies. Key ingredients of the CEM are (i) solving a zeroth order problem whose solution is determined up to certain amplitude functions; (ii) assuming an expansion for the solution all of whose higher order terms depend on (slow) time through the amplitude functions only; (iii) assuming that the right sides of the equations of motion for the amplitude functions are expansions whose coefficients are functionals of the amplitudes. These coefficients are determined by appropriate solvability conditions for the hierarchy of linear equations resulting from insertion of all these assumptions in the original equations. Collecting the desired number of coefficients, we obtain approximate equations of motion for the amplitude functions as the result of the method. I believe that the techniques explained in the present paper will be useful in many other problems of physical interest.

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